Introduction to Quantum Mechanics: Key Points

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1. "Particles" and "waves" are classical concepts.
   In Quantum Mechanics we think of everything in terms of

   $$\boxed{\text{Particles described by wavefunctions.}}$$

2. Every observable (i.e., thing you could measure) Q has a corresponding operator $\hat{Q}$ that acts on wavefunctions.

	classical observables	$\rightarrow$	QM operators	
e.g.	momentum	$\rightarrow$	$\hat{p} = -i\hbar \dfrac{\partial}{\partial x}$	
	position	$\rightarrow$	$\hat{x} = x\,\times$	
	energy	$\rightarrow$	$\hat{H} = \dfrac{\hat{p}^2}{2m} + V(\hat{x})$	(for a nonrelativistic particle)

   where the energy operator $\hat{H}$ is also known as the "Hamiltonian".

3. The time evolution of a particle's wavefunction $\psi(x,t)$ is given by the time-dependent Schrödinger equation

   $$\hat{H}\,\psi(\vec{x},t) = i\hbar \frac{\partial \psi}{\partial t}(\vec{x},t)$$

   Using the known forms of $\hat{p}$ and $\hat{x}$, we can write the time-dependent Schrödinger equation for a nonrelativistic particle of mass m in potential $V(x)$ as

   $$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{x},t) + V(\vec{x})\,\psi(\vec{x},t) = i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t).$$

4. For a free particle $V(x) = 0$ so we can solve the time-dependent Schrödinger equation to find the wavefunction of a free particle. The solutions are *Unnormalizeable* wavefunctions called *plane waves*,

   $$\psi(x) = A\exp\bigl(i(\vec{k}\cdot\vec{x} - \omega t)\bigr)$$

   where $k = 2\pi/\lambda$ is the (angular) wave number and $\omega = 2\pi f$ is the angular frequency.

$\hbar k = p = \dfrac{h}{\lambda}$	Momentum $\leftrightarrow$ wavelength
$\hbar\omega = E = hf$	Energy $\leftrightarrow$ frequency

Requiring that this wavefunction evolve in time according to the Schrödinger equation, we find the energy-momentum relation for a free particle, $E = \dfrac{p^2}{2m} \quad \Rightarrow \quad \omega = \dfrac{\hbar k^2}{2m}$

5. We will deal with two types of wavefunctions, Normalizeable and Unnormalizeable.

	Normalizeable	Unnormalizeable
Possible physical state of a particle?	Yes	NO
Normalization condition	$\int_{-\infty}^{\infty} \lvert\psi(x)\rvert^2\, dx = 1$	None

   Unnormalizeable wavefunctions are useful as building blocks for constructing physical, normalizeable wavefunctions.

6. When you measure some observable property of a particle, its wavefunction may not predict a specific value for the result. In general a wavefunction only tells you the probability distribution $\mathrm{Prob}(Q)$ for getting the answer Q when you make a measurement.

   So far we only know how to calculate the probability distribution for *position* measurements, where the probability to find the particle in the interval $[x, x + dx]$ is

$$\mathrm{Prob}(x)\, dx = |\psi(x)|^2\, dx.$$

7. Even without knowing the full probability distribution, we can still obtain the "expectation value" $\langle\hat{Q}\rangle_\psi$ of an operator $\hat{Q}$ in a state ψ. This is the average of the values obtained when measuring Q on many members of an ensemble of systems, each of which has been prepared in the same state ψ. For a normalized state,

$$\langle\hat{Q}\rangle_\psi = \langle\psi|\hat{Q}|\psi\rangle = \int_{-\infty}^{\infty} dx\, \psi^*(x)(\hat{Q}\psi)(x)$$

   For an unnormalized state,

$$\langle\hat{Q}\rangle_\psi = \langle\psi|\hat{Q}|\psi\rangle = \frac{\int_{-\infty}^{\infty} dx\, \psi^*(x)(\hat{Q}\psi)(x)}{\int_{-\infty}^{\infty} dx\, \psi^*(x)\psi(x)}$$

8. The "uncertainty" in observable Q in state ψ, written ΔQ_ψ, is the standard deviation of the probability distribution $\mathrm{Prob}(Q)$, so

$$(\Delta Q_\psi)^2 = \langle\hat{Q}^2\rangle_\psi - \langle\hat{Q}\rangle_\psi^2$$

9. Heisenberg's Uncertainty Principle states that for any state ψ, the uncertainty in position and the uncertainty in momentum are related,

$$\Delta x_\psi \, \Delta p_\psi \;\geqslant\; \frac{\hbar}{2}$$

I.e., wavefunctions with a small uncertainty in position must have a big uncertainty in momentum, and vice versa.

10. The commutator of two operators $\hat{A}$ and $\hat{B}$ is the difference between acting with $\hat{A}$ first and then $\hat{B}$, versus acting with $\hat{B}$ first and then $\hat{A}$,

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$

Using their explicit forms we can see that

$$\begin{aligned} [\hat{x}, \hat{x}] &= 0 \\ [\hat{p}, \hat{p}] &= 0 \\ [\hat{x}, \hat{p}] &= i\hbar \end{aligned}$$

so the momentum and position operators *do not commute*. This is the origin of the Heisenberg Uncertainty Principle.